PRIVATE UNIVERSE PROJECT IN MATHEMATICS WORKSHOPS

PROBLEMS AND POSSIBILITIES

WORKSHOP 6: POSSIBILITIES OF REAL LIFE PROBLEMS

PART 1 - THE "CATWALK": REPRESENTING WHAT YOU KNOW

NARRATOR: During a trot, do horses ever have all four hooves off the ground?

More than a century ago, a wealthy Northern California businessman, Leland Stanford, enjoyed running horses on his private track. After an argument with a skeptical friend, he placed a bet claiming that there are times during the trot that horses have no hooves touching the ground.

To settle the wager, Stanford hired Eadweard Muybridge, a San Francisco photographer. Muybridge rigged a series of still cameras in a line, so that the shutters triggered in sequence as the horse ran past. Stanford won the bet.

Muybridge continued this work and in 1887, he published a ten volume set of animal and human motion studies, called "Animal Locomotion".

Muybridge's photographs opened up a whole new branch of scientific study... If they were the focus of a situation where students were invited to build important ideas about motion and change, would they also trigger the need for calculus?

CHUCK WALTER: Looks like every little red light in the room is on, so I suppose that means that we're ready to go.

NARRATOR: In July 1999, before their senior year in high school, 18 students attended a two-week Summer Institute at Brearley High School in Kenilworth, New Jersey.

At some point in their education, all of these students have been involved in the Rutgers long-term study - some going as far back as first grade.

A few were part of the Kenilworth focus group. Others traveled to the Institute from New Brunswick and other districts.

CHUCK WALTER: We're going to think about something that's a little bit new, I hope.

NARRATOR: A few years before the Institute, two mathematicians at Brigham Young University, Bob Speiser and Chuck Walter, were invited to begin discussions about teaching calculus to all entering freshmen. What would the course look like? What activities would it include? How could it be relevant -- not just to math or science majors -- but for all students?

CHUCK WALTER: So, let me pass you or ask people to pass you a page which has some photos on it.

NARRATOR: One activity they looked at was the "catwalk", based on Eadweard Muybridge's photographs of a cat.

CHUCK WALTER: Just look at this for a little bit if you will. These are 24 pictures, 24 frames of a cat in motion. Do any of you have cats, by the way?

NARRATOR: Photographs are the only evidence that exist for a real-life event - a cat's walk across a photographer's set that took place more than 100 years ago. The photographic evidence is not clear-cut. Open to many different interpretations, the cat problem itself might be more typical of the real-world problems that the students will soon solve in scientific or business careers. The risk for the researchers was that the students might fail. On the other hand, success could help make calculus accessible to more students.

CHUCK WALTER: ...there's a backdrop to the cat, kind of a grid. The dimensions of that grid, each line is five centimeters apart. You see that indeed these photos were taken at an interval of 0.031 seconds a piece. Angela, what was your remark about the time frame?

ANGELA: 0.71 seconds, so it's not that long a time.

CHUCK WALTER: Not even a second. Let's think about the motion of this cat and in some sense, how we might begin to describe it. We have a particular set of questions that we would like to use to start this discussion, and then we can like to begin to think about the cat, to try to get a sense of how you're starting to think about it.

NARRATOR: The problem the students had to answer was: How fast is the cat moving in Frame 10? and, How fast is the cat moving in Frame 20?

ROBERT SPEISER: It was the first time with high-school students. And it was also the first time we'd ever attempted it with students who hadn't studied calculus yet. For us this was a complete adventure - what would happen? It was wide open, and in fact we were very nervous about it. Suppose the whole thing just fell on its face, suppose they got nothing? Suppose they got frustrated? Suppose they got the feeling that this was beyond them? Suppose we didn't learn anything?

CHUCK WALTER: One of the very interesting things in this sequence of photos is in Frame 10, there's a huge discrepancy in the average velocities of the cat on one side of Frame 10 and on the other side of Frame 10. so, the question comes up - Just what can we say about the velocity of the cat at Frame 10, and how would we begin to think about this?

ANKUR: How many centimeters did you say you moved? Like, three?

JEFF: I said two.

NARRATOR: The students' work was documented by four video cameras and more than a dozen researchers.

ROMINA: Between 9 and 10 there's , like, no difference. There's like, a difference I can't measure.

NARRATOR: The materials the students had to work with were the photographs, overhead transparencies, rulers, pens, paper, and a graphing calculator.

MAGDA: I wonder if we just measure how much distance he covered from this picture to this picture and divide it by .031, right?

ROMINA: The cat's going different speeds every picture.

MAGDA: Is he really?

ROMINA: Magda!

JEFF: You would think it would be that easy.

NARRATOR: Finding the average velocity over a given interval of time is something many students are familiar with from middle school or even earlier. These students, all of whom have completed at least two years of high school math, might think the question is very simple: Velocity equals Distance divided by Time.

However, Bob and Chuck have challenged the students with a much more difficult question: How fast might the cat be moving in Frame 10, at that single moment?

Measurements made by the students will show that the cat was going about three times as fast between Frames 10 and 11 as it was between Frames 9 and 10. Will the students be able to say anything for sure about what happened in between?

JEFF: Within the first 10 minutes, everyone has an answer. And in the first 20 minutes, everyone scrapped their answer and is totally lost. And I mean, this was a classic Rutgers kind of situation: We went in; we had the two main components in finding out the speed of velocity, and we went in and we all scribbled it down and we were like, all right, that's it, what's next? And then we start talking and then one of the people would bring something up and we'd be like, "Oh, we didn't think about that." And then as you start looking into it, you've kind of got to push what you did away and you start over and you get into some different things and see where it would take us.

MILIN: I was thinking we should also do it from 10 to 11.

MICHELLE: That's going from 9 to 10.

MATT: That changes things. That changes a lot of things.

ROBERT SPEISER: One of the things we were learning in our experimental teaching in the calculus course at the university was that a lot of care and time needed to be invested in what looked like very basic activities: making the actual measurements, going from those measurements to calculations of average velocities over the intervals from one frame to the next. That very basic work had to be done, and discussed, and presented and worked through.

MAGDA: This is what we did, we measured each box, and it's like .36 cm and it equals 5 cm., right?

CHUCK WALTER: You physically measured each box on the page?

MAGDA: Basically, no, I measured from this line to here. And it was like 3.6.

CHUCK WALTER: Oh I see. So you measured it though?

MAGDA: Yeah, and divided it by ten. I got the average.

CHUCK WALTER: OK. I see what you're saying.

MAGDA: Each box is .36 cm. But in actuality it's 5 cm. You know what I'm saying?

MATT: I was thinking we should also do it from 10 to 11 because it's kind of moving a little bit faster in 10 than it is in 9...

ROBERT SPEISER: Somewhere in the elementary school experience, questions about speed, distance, and time begin to come up. A car is going at 45 miles an hour: How long does it take for it to go a certain number of miles? Kids need opportunities to think about that kind of problem, because I think it gets very deep into what multiplication and division get at. So, right there, the seeds are being planted for this later, much more detailed study of motion, in which now we're going to look at changes of speed.

MICHAEL: ...That's what it says. From here to here, it's going four centimeters a second. Does it make sense to you? It doesn't make sense to me. But right there, he's moving four centimeters a second. It doesn't make sense because from here to here he moved two.

__: Right. Centimeters.

MICHAEL: Oh, he's moving two, and that small amount of time, he'd have to be moving 64 centimeters a second, so it doesn't make sense.

__: Isn't that an awful lot?

MICHAEL: Well, if you think it moved two centimeters, which is not that big, but in .031 seconds, which is a very small amount of time, and 64 is what - like this big? But in a second? It just doesn't fit.

ROMINA: Well, what we did is we went frame by frame.

MAGDA: That's a good idea.

AQUISHA: ...(inaudible) right over the same one. Look.

AQUISHA: At my table, there is about four kids that have been like best friends - and there are six of us - since first grade. And that was a little intimidating to me and I sat there really quietly, just trying to figure them out and trying to get in the conversation somehow. And the first day, I was extremely quiet, but after a while, they made me feel a lot more comfortable. And I think it's easy to work with them because I find that they're open-minded and if you have something to say, they listen. We listen to each other and we work things out.

ROMINA: Okay, we worked up to 10 and then we just trash our idea.

AQUISHA: I wish these were different colors though... We can color in all the cats like green.

ROMINA: Yeah, why don't we - we could do it on the overhead. So we could see it better.

AQUISHA: But I mean like, when we put them on top of each other, they're both black, so it's hard to even tell. I'm going to try that. I'm going to color them all in like red or something.

NARRATOR: One of the challenges the students faced was finding a way to take measurements on the photographs. Aquisha invented a method of lining up each frame with the previous frame and marking the distance directly. Next, they organized these measurements into data tables.

ROMINA: It is in centimeters, but -

MAGDA: We have two, so .1 centimeters - you've got to change it into the actual measurement.

ROMINA: Yeah, this is like actual. It is actually what it is on our paper.

AQUISHA: Oh that's what I wasn't getting...

ROMINA: Yes, so it can't equal 5 cm. because it's not in proportion. We haven't put it in proportion yet.

MAGDA: So we've gotta put it in proportion an then the speed will be right...

ROBERT SPEISER: For us, a representation is a way of showing information. There's a tendency to think of math problems as things that are done wordlessly in the hand or wordlessly on paper with just figures or symbols. But even thinking privately, we're using the paper as a kind of memory device or maybe more than that, an actual medium for thinking. Then when we work publicly, we're trying to explain why we think that something is true. How can we bring evidence forward without using a representation of some kind? I don't know how we can think without representation.

NARRATOR: If representations are the currency of mathematical discourse, then this activity would be rich in different ways of looking at the problem.

JEFF: ...How many things over is it?

MICHELLE: Ten.

JEFF: It's 10 cm from the first to the end.

MICHELLE: 10 cm. two things.

JEFF: From here to here?

JEFF: Which line is this? Is this from the end of the 10 or the end of the 5?

MICHELLE: That one is the end of the 10.

JEFF: So he moved about maybe 2 cm from 9 to 10. If that's the case, dude, he's moving ridiculously slow. If we just show from 9 to 10. You know? I mean, he's like barely creeping.

MILIN: Yeah, but you have to divide by .031, so that number will get a lot bigger.

MATT: He moved nine centimeters in less than a second, basically.

JEFF: Over here he moved -

MATT: - over 100 centimeters. This is all within .71 seconds.

JEFF: Yeah. You know what I'm saying? Back here, dude, he's merely crawling. He's barely moving.

MATT: Yeah, he like just shoots out...

ANGELA: ...He is barely moving.

MILIN: He is barely moving. It's .031 seconds...

MICHELLE: Right now my group has three different answers. We don't know if our answer is right. We don't know if the way we're doing it is the average velocity from the beginning to the time frame or if it's the velocity at that instant at that time frame, and that seems to be the major problem right now.

ROBERT SPEISER: If you want to figure out what's going on in Frame 10, you have some choices. For example, should you work out the average speed from Frame 1 through Frame 10 and then the average speed from Frame 10, say, to Frame 20 and then compare those? Or should you look just locally, right near Frame 10 - at what happens from Frame 9 to Frame 10 and then from Frame 10 to Frame 11? Why would you prefer one over the other? Gradually in their arguments, they zeroed in on the local information around Frame 10. I think that was a huge step forward.

CHUCK WALTER: Okay, can we let this group come to the board up there and tell us what their thinking is?

NARRATOR: At the end of the first day, Chuck asked the groups to summarize their progress. Starting from the location of the cat's nose in Frame 1, Michael's group made this graph of the distance that the cat traveled, plotted against time.

MICHAEL: Well with this graph, it's not the velocity, it's just the change in distance. But what I see here is a certain velocity and another one. So, the time would be as you go along, you know, and the distance is the difference between each and every one. So, that means he's walking at a constant rate for a very small amount of time.

Then when this changes, that's around when he starts crouching down and he starts walking - and 10 is around here and he starts going at this speed. This is not accelerating here. He's actually going the same speed. That's why all these guys are like the same apart - thatís basically a line. It shows two lines.

It may be a little off in there, but basically what it is - he's walking in the beginning going this speed and running at the end, going a faster speed.

MICHAEL: In a traditional way, a teacher would bring out an idea or like something like that, but in the Rutgers way, a kid would come up with something and show it to the class. With the cat thing we're learning, either me or Matt or Romina's group would just go up there and we would tell each other - "Well, this is my graph and this is what it shows. I'm teaching you about what I discover, you know?" And like, that wouldn't happen in a regular class. By us going up there and showing us what we got, it helps other groups use that information to finish their task or whatever.

[MUSIC]

NARRATOR: The next day, the researchers asked the students to continue their presentations.

ANGELA: Okay, what we did is we did the average. We measured the velocity between the 9th and the 10th frame and then the 10th and the 11th frame. So, we figured that if we took the velocity between these two frames and these two frames, and then figured out the average, it would give you like the middle, which is the 10th frame.

NARRATOR: Angela & Shelly's group came up with precise numbers for the velocities in Frame 10 and Frame 20.

ANGELA: And we got 145.161 centimeters per second for the velocity. Then we did the same thing with the 19th and the 20th and then the 20th and the 21st and we got 354.839 centimeters per second.

CHUCK WALTER: Let's see. Oh, we were going this way, weren't we? Kind of clockwise.

NARRATOR: Romina transferred the data from her handwritten table to her graphing calculator. This allowed her to share two more representations with the class.

ROMINA: This is our time versus distance. At first it's going slower 'cause it's walking and then it accelerates. It's going more distance because it's like running, galloping, whatever. And then we also graphed our distance versus speed, and it's a little different from Mike's because it shows, it shows like the cat... We kind of agree with

Mike at the first part where it's walking, it's going at like a constant rate. But then it shows when it starts like its gallop, it kind of like pauses a little bit and then it starts accelerating faster and faster until it reaches like a climax and then it starts coming down. So, towards the end, that's where the cat's stopping. Which Mike's didn't show that; it kind of just showed like a constant rate at walking and running. So, ours just kind of disagrees with theirs a little. That's all we did today. Aquisha, you want to do your thing?

NARRATOR: Building on work from the previous day, Aquisha came forward with a powerful representation: one that changed the way the entire group approached the problem from this point on.

AQUISHA: This is what I did yesterday. By putting the two of these on top of each other, like Frame 1 over Frame 2, you see the distance that the cat traveled. I measured from the nose here to the nose of the next one. I used the lines, the thick lines, to line them up.

NARRATOR: Building on work from the previous day, Aquisha developed a unique way of representing the movement of the cat. Alternating two colors of marker, she traced the distances the cat ran between frames as short line segments lined up end to end.

AQUISHA: This represents, from frame to frame, what the cat traveled. Like this one's from Frame 1 to 2 and then from 2 to 3, up to 24. And today, I put them all together. This is just all the distances. I connected them.

NARRATOR: Aquisha's line representation clearly shows the central issue of this problem: there is a big change in the cat's velocity, right around Frame 10.

ROBERT SPEISER: The representations that Aquisha is showing us were entirely her own. She's building from the same numerical database that Romina and Magda are working with. But she's representing the information the numerical information in that database in a very strikingly different way, in a much more visual way. A much more personal way.

CHUCK WALTER: Thatís a remarkable idea. What kind of sense of the cat's motion can you see here? Can you see anything at all? What do you think?

BENNY: I guess you see the total distance traveled by the cat.

AQUISHA: You can see the difference that the cat was walking slow and how it got faster.

JEFF: You can see how, kind of like Mike was arguing the other day, how he's going the same speed a lot in the beginning, you see a little time where he's accelerating, and then he pretty much runs along the same at the end.

CHUCK WALTER: Does that picture that Aquisha's presented there carry the same sense that some of the other graphs that we've seen does? Could you say why, how?

BENNY: I think Jeff said you can see how in some areas the speed stays at a constant rate, then changes slightly, then as it changes it stays at a constant rate. You can kind of see that from that.

CHUCK WALTER: How would you see that, Benny? Tell me a little more about what you see...

ROBERT SPEISER: Students find the standard scientific representations such as tables and graphs a lot more difficult than we often think that they do. It seems that unless they have the opportunity to build personally meaningful representations, whether they're standard ones or not, and work through what the information is that is carried by this representation and how to read it, then the really important standard scientific representations, like tables and especially graphical representations, that are so important remain difficult, remain problematic for them. Aquisha's, I think, is a really beautiful example of how just wonderfully creative and powerful a personally-developed representation could be.

AQUISHA: The first dot represents from one to two.

BENNY: Okay. So, I mean five. So, after five, it begins to accelerate.

CHUCK WALTER: Thanks, Benny.

ROBERT SPEISER: In a sense, Aquisha kicked open the door, because we need to understand how the cat was moving in space, and laying it out along a line is the first step to seeing that. It was delightful to see this, partly because of the visual power of this simple, modest looking image, and the effect it seemed to be having on the other students. Then the next step, of course, was to encourage them to go further, to build this in space, you know, in a much larger frame, in a frame that they could physically move through...

CHUCK WALTER: I've got an idea. Be a cat, okay? I'm wondering if, if in fact we can do this, if in fact we might be able to be this cat. If we think about this, this is how the cat moved, right? We see not only how far it went, but we see kind of spacings of distances. Isn't that what this represents, Aquisha?

What if we, instead of marking this out here like this, what if we marked this out on say the floor? You know, maybe expanded it so that it would be not just nine centimeters long or 17 or whatever. We would make it, make it big enough that we could put it out here.

ROBERT SPEISER: Mathematical knowledge grows from our actual experience of moving around and living and thinking about how we operate in the world. What does it mean to get at the story of what the cat did? How would you understand it? Would you describe it for me outside, in that we have these numbers, that these are our measurements, these are the speeds, that it fits into this graph? That's one way to do it. That leads you to one set of representations. Another way to do it would be to try to do a movement that would be like that. What would it mean to do that? How would you get at the story? That leads to another set of representations.

CHUCK WALTER: We could mark the frames, right? Like Frame 1 here, Frame 2 here, just like Aquisha did. And then maybe we could see if we could move like the cat. Is that a dumb idea or do you want to try it?

STUDENTS: Let's try it.

NARRATOR: **After some discussion, the students selected a scale of 10 times the distances the cat ran. The runner's task was to pass each mark exactly on the beat of a steady drum, in effect, to "be" the cat, but on a human size scale.**

[Drum beating]

CHUCK WALTER: All right, Benny wants to go. Ashley, hang on to what you were thinking about when you went through 20, okay? [Drum beating]

ROMINA: The actual size of the paper layout was 130 centimeters. Then we made it a little bit bigger in the library, but you couldn't feel the full effect because like at first, in the span of two feet was the first ten intervals, so you couldn't really do it. So, we figured we'd multiply it by 50, that way we'd actually have to be walking - from a piece, we marked the piece of tape - from like tape to tape, we'd actually have to be walking and then running.

NARRATOR: On Day 3 of this activity, the students chose to work on a still-larger scale, one that is fifty times the distance the cat actually ran.

ROMINA: Should I write-

MAGDA: You can write the distance in Frame 1.

ROMINA: Should I write Frame 1? Like, I'll put one here and then two?

MAGDA: Yeah.

MAGDA: 375, Romina.

ROMINA: Isn't there a spot here where it's like the same thing over and over?

MAGDA: Yeah - one, two, three, four, five.

ROMINA: 375 is the one?

MAGDA: Yeah, 5 times 375.

BENNY: This is basically going to increase our stride so it makes it look like we're picking up speed more, you know, instead of just walking like at a steady pace, so we can see it more. Like more visibly we can see it, instead of just walking and it looks like we're just taking a step. See, with this, we're taking longer strides and we can actually see it. So, that's basically it.

[Clapping]

MICHAEL: It looked like they were going slow in the beginning, and they stopped - like slowed down real quick, you know, when the things get really close together, and then they start to pick up speed until they start to run. I think they felt like at the end like they weren't speeding up - they were just running - at a constant speed, and it was easier to see exactly what those dots on the graph really meant.

ROBERT SPEISER: How do those scientific representations come to life? Well, it seems it's pretty obvious when you think about it, although our educational culture makes it look so foreign: get in there and move! And then connect that with the numbers, with the graphs, and with the other pictures, and let's see what kind of sense we can make through connecting all these things.

ROMINA: It made sense of it. Like, we had the numbers there, but they didn't make sense to us. We didn't understand how something could change speeds so fast, what was going on, why it would change speeds so fast. But when you do it like a real life version of it, you can see what the cat's doing so you understand it. And like our graphs would go up all of a sudden and like fly down. But then when we did it, we could see like it was accelerating, it came to a peak, and then it was slowing down. It just made sense of all the math.

[Clapping]

NARRATOR: How might the personal experience of running help these students to deepen, organize, or clarify their growing understanding of the motion of the cat?

PART 2 - BETTING ON WHAT YOU KNOW

NARRATOR: After running the model in the hallway, the students began an open discussion about the cat's motion.

CHUCK WALTER: I was wondering what kinds of questions or what kinds of responses came out of this morning's little workout there. Let's be kind of brutally honest with each other and bring real questions out on the table and then we can address those. Jeff?

JEFF: On the brutally honest bit, I'm just a little confused. I thought it was good to make a model. I think that's always good to do something, like if you can do it physically yourself, but I don't see what we're getting from doing all of it.

CHUCK WALTER: That's a pretty brutal question, right?

JEFF: Yeah.

CHUCK WALTER: But I think it's also right to the heart of the matter and the point. Okay?

ROBERT SPEISER: I think Jeff's question is a good question. Some work still needs to be done to connect what was done in the hallway with the graphs and the tables and the other representations, including Aquisha's, that had been developed before.

MATT: I just think that we finally figured out, like what Mike said, how he thought the cat stays the same speed or whatever. The cat doesn't stay the same speed. It's constantly speeding up and speeding up and speeding up.

VICTOR: And the way I see it is I think that the graph, whatever, that this group right here came up with - Romina - how they showed the change in speed and position in time and all that stuff. I think whoever ran, they can kind of feel it, because in the beginning, they were running at a constant pace. Then they kind of slowed down; then they picked it up, picked it up and kept on picking it up, which was what they showed.

CHUCK WALTER: So, what you're saying is kind of adding to what -

VICTOR: - He said, but also proving their point with their graph.

CHUCK WALTER: That was an interesting graph. I'd like to take a look at it again sometime if you guys still have it.

ROMINA: I'll check. Hold on.

CHUCK WALTER: Anything else? I want to hear what Benny says because Benny's been a real spokesperson for a lot of this over this period of time.

BENNY: Well, I was just saying how this model and with the model outside, I was thinking it gives us more of a visual explanation and not like a mathematical explanation. You know, like you can actually see like between point 10 and 11; this is where it picked up in between points 19 and 20; this is where the speed begins to get fast. Just something you can see: All eyes, all eyes on the cat.

ROBERT SPEISER: It seems to me that Benny is trying to bring the discussion very strongly back to what is the cat actually doing. I think the "seeing" is a little deeper than literal, physical seeing. He's going for the abstraction that's behind the different representations, that no one representation captures the idea of what's happening, but somehow we need many of them.

NARRATOR: Romina re-displayed her graph of Velocity over Time.

MICHAEL: Your time was - how does it go up, from .031 to .062?

ROMINA: Yeah.

CHUCK WALTER: And is that time? It looks like time across the bottom.

ROMINA: The x is time and the y is velocity.

MATT: Each of these little dots here, these are all his velocities at a certain time. So, if you see how the line goes, that's his climb in velocity. Yeah, this is like his acceleration. You can tell, like from here to here, his acceleration goes down; and from here to here, it starts to skyrocket up like that. Then he evens out for a while; then it goes down a little bit; then it skyrockets up again.

CHUCK WALTER: Where is the acceleration?

MATT: These lines. This line here, this line here... Where it starts to swing up.

ROBERT SPEISER: Matt had a graph of velocity or speed plotted against time. He was referring to it as an acceleration graph. So, Chuck's question is inviting him to explain where he saw the acceleration, and then he does it splendidly, by looking at where the velocity was changing the most rapidly.

MAGDA: Where is he going at a constant speed?

MATT: At these ones here, he's going at a constant speed.

ROMINA: The photographer couldn't tell the cat, "Okay, you're going to start walking; then just speed up into a run." So, the cat was probably walking and then he did something to like it make it, like scare it. And then all of a sudden, it went from like a constant walk, just like regular walking, to like speeding up into running, because something had scared it and made it move.

VICTOR: Right, right, right. What I see, right, is like these constant speeds, it's like the first constant speed, what I figure is that he's most likely to just walk. But then at the second constant speed, I think, you know how he just shoots up for a quick minute at that first "A" that Matt has drawn up? I think that's when he jumps up. Then while he's in the air for that like split frame right there, he's in the air traveling and that's constant, and that's when he kind of comes down, like after the constant speed that he separated, and then he just starts walking again. That's what I gathered, I think he just stops because he doesn't keep going.

MATT: I'm not even sure - how many steps does that cat take? Because this here, this might be when he pushes off and what speed he gets to. and when he pushes off. And then this would appear to be what speed he gets to, and then he would push off again. That's what I'm thinking too.

ALICE ALSTON: Oh, what, though, you were looking at our pictures?

BENNY: Yes.

ALICE: How many steps did you think the cat took?

BENNY: Like four or five, maybe four or five. Because at the end there - Somebody got the thing?

MATT: That's like four or five because -

BENNY: Yeah, that's like four or five. Now if you look at the first step, he begins his step right here. And if you look, he's lifting his leg up gradually, gradually, gradually, gradually. He begins to put his foot down here and then his foot touches the base right here. That's one step. Now, he begins another stride right here, gradually lifts his back leg up. You can see - this is the foot we are going to judge it by - his back leg, when he lifts his back leg up again and he lifts it up high, high, high, then he begins to bring it down, touches base again - that's the second step. Then as his speed picks up, it starts to take him less time to take a step. So, in this third one, his back leg - touches base here and then he begins another stride right here.

MATT: Basically he took like four steps.

BENNY: Four steps.

MATT: When he's walking, you don't see that acceleration; like, you're not going to see it on this graph. Like, this is his first two steps here, while he's walking at a constant speed. And all of a sudden, on that third step, he gains his speed, and this is when he lands, and then that's his speed again, when he pushes off.

CHUCK WALTER: This is really getting interesting. It looks to me like there are some different kinds of things entering the discussion. First of all, we have - you produced some data; you produced some graphs, lots of different kinds of graphs. But there's also I think in the mix now, the ideas that are starting to come out of our walking/running cat experience. And of course, there's still the photographs.

ROBERT SPEISER: Romina began this series of connections between the cat's actual movement and the velocity graph. Several of the other students, culminating with Benny, developed that idea. So it's the first time that the idea of energy has come into the mathematical discussion. Where was the force applied? You can see that in Muybridge's photographs.

NARRATOR: The next morning, Day 4, the researchers posed a hypothetical problem, with the students' math teacher, Ralph Pantozzi, pretending to be the CEO of a large corporation.

CAROLYN MAHER: Let me say a few words. We have a 10:30 deadline. But I'm a little concerned....

ROBERT SPEISER: Getting down to the bottom line - whatever you say, how you justify it? For the last 10 years, these students have been part of a study of justification, explanation, and proof. And it's an invitation for the whole community to fit some reasoning together, and to present it.

CAROLYN MAHER: ...But I'm really concerned because my job is on the line. Mr. Pantozzi wants a report at 10:30 and he put me in charge of this project. You're all the wonderful consultant staff that has been working to help me on this. So, I'd like to hear- I mean, I don't know what's going on. I've been called from meeting to meeting; I've been really sort of distanced from what you've been doing this morning. Why don't you tell me what the problem is exactly, because I'm not even so sure I understand what the issue is here.

VICTOR: He's making his wager on what the speed is at Frame 10 and what the speed is at the 20th frame as well. But after we did our calculations -

CAROLYN MAHER: Hold on a minute, Victor. The exact speed? How accurate?

VICTOR: How accurate? Well...

CAROLYN MAHER: To the nearest tenth, hundredth, thousandth, millionth?

VICTOR: We calculated it to three places over the decimal.

MATT: Mr. Pantozzi didn't specify.

VICTOR: Yeah, but he's going to -

MATT: Yeah, but if he wants results, he's got to specify what he wants. We can't -

CAROLYN MAHER: Well, he's going to make this wager or not make this wager, and he's- How right does he have to be? We have to advise him?

VICTOR: Now, when you listen to our statement, then you will understand. We wrote, we believe that this would be the best position to take on the cat's motion. However, we believe that this is an approximation rather than an exact answer. We will not be held responsible for any monetary losses. [laughter]

NARRATOR: Each group came forward to present their findings. Victor's group presented the same velocities for Frames 10 and 20 that they had shared on Day 2 of the investigation.

VICTOR: Okay, now, what we believe is that you can actually tell the velocity of the cat at Frame 10 and 20, but you don't necessarily know how accurate our numbers are because everything we had to do, we kind of like eyeballed our measurements and this is not a real life scale. So, we can't really get as accurate as we want. What we did was, we used the basic physics formula of 1/2 times the quantity of the initial velocity plus the final velocity, and we have a velocity of about

145.161 centimeters per second and give or take 20 centimeters/second due to inaccurate measuring. And for Frame 20, we have about 354.839 centimeters per second. If you look at the picture, you see that about Frame 20, he's really taking off at almost twice the speed because he's springing forward from the gallop. So, you can see actually the change in speed being that drastic from Frame 10 to 20, so that's why our numbers, you know. But, we do not suggest that he make the wager, however, because gambling is illegally and morally wrong. So there you go. Any questions anybody?

ROMINA: Yeah, I've got a question.

VICTOR: I thought you would.

ROMINA: When you say you take the average, right, you're saying you start off Frame 10 with the 2 - it's not the velocity - whatever that 2 divided by .031 is, right? And you end Frame 10 with 7 divided by .031. So, that's a really big jump. How do you - when you take the average, where are you saying? Is that like right in the middle of Frame 10, that's the speed it's going?

VICTOR: Right, right here? Exactly. Because from Frame 9 and 10, it's moving - the change from 9 to 10 is about 2 centimeters and from 10 to 11, which is the next frame over, it's 7 centimeters. That's just because of what the movement of the cat did.

ROMINA: So, in Frame 10, though, he's going like -

VICTOR: Halfway, something like-

ROMINA: But in Frame 10, he's changing speed, so he's going, he starts - In Frame 10, he has to somewhere - up from like 80 to like - our measurement said 80 centimeters per second to like 200 centimeters per second, when we divided our numbers. So, that's like a big gap. Like, you can't - Why did you take an average on that? You don't know exactly where he's taking the - Like, I can understand in the second one, the average is a little bit more like realistic; like you could bet on that one. But the first one - you can't bet on that number because he's going so many different speeds in that one frame. You understand what I'm saying?

MATT: Yeah, we can't bet on it.

ROMINA: Yeah, exactly.

VICTOR: That's why we try to take an average, you know, a guessed average. I mean, that's why we said that it wasn't really, really accurate, especially since the frame's before that, he was like doing all these other things. But the second one, we can't bet on it.

MATT: We wouldn't recommend he bet on it. I wouldn't bet on it.

ROMINA: Thank you.

VICTOR: No more questions? Okay then. I guess we're done.

TIM SWEETMAN: It sounded to me like your recommendation was don't bet, right?

VICTOR: Right, don't bet. I mean, if you had to be real exact, I'd tell him not to wager.

JEFF: If it just had to be to the second, would you tell him to bet?

VICTOR: If they had to be to the second-

JEFF: You think he's going to land in there around three or whatever the number is?

VICTOR: No, not really. When you have to be exact, I don't think you should bet on it.

NARRATOR: Shirly and Mike's group presented next.

SHERLY: Okay, what we did was - Because originally, we measured from the cat's nose or whatever, but we took a point in the body that doesn't really move that much, so we took the point for the nose and the point for the base of the tail and we averaged it together for

Frames 9, 10, and 11. And then he found the difference between the distances and he got the velocity by dividing by the time and he got those numbers and he averaged those two together and he got the average speed, basically for Frames 10 and 20. But they're not accurate, so we don't recommend him betting.

MICHAEL: Don't bet.

TIM SWEETMAN: That's it? Don't bet? Well, we already bet.

MICHAEL: Since all three groups did this and got three different answers, I'd say don't bet. Because if we all got the same number, like 149 point something, go for it. But if it's like 160, he got 80 or something, and we have 140, just don't.

ROMINA: Yeah, we all had the same process, but like different-

TIM SWEETMAN: How could you have the same process and get 80?

ROMINA: Because our measurements are a lot different than theirs. And we eyeballed it on and they used a ruler.

MICHAEL: I don't know what to make of this, but I would say don't bet.

TIM SWEETMAN: All right, let's hear what this group has to say. I want to see where this 80 came from. It bothers me a lot that we're so far off. He won't be happy with me when I tell him this.

ROMINA: What we did is we started off with like measuring like with our ruler and like where the cat's nose moved, but then that wasn't working for us because the measurements were getting too small, so then we kind of eyeballed it. And we said from 9 to 10, he moved about half a box, so that's 2.5 centimeters. And when we divided that by .031, it came out to 80.

TIM SWEETMAN: You used the nose?

ROMINA: Yeah. Our numbers go from 80 to 200; that's like a big difference. So, if you take an average - like, we just said we wouldn't bet on an average.

ANGELA: Frame 10 is right in the middle of those two, so it's going to be the average.

ROMINA: Yeah, but in Frame 10, he's going-

MATT: It doesn't mean the average.

ROMINA: There's a lot going on in Frame 10. I just don't think he should bet on Frame 10. Frame 10 is too like - Frame 10, too much is going on in Frame 10. Like, like 'cause if you're going like technical, the guy could be like, "Well, I meant at the beginning of Frame 10 what his speed was," or the guy could say at the end of Frame 10.

ROBERT SPEISER: What we're sure of is that there was a large change in the velocity. What we can't be sure of is what that velocity might have been just at that moment when Frame 10 was exposed. It's a very hard thing to talk about. It's not the speed, somehow, that we can lay our hands on around frame 10. It's the fact that the speed is changing so much. And yet, we don't know the speed at frame 10. And probably we can never know it.

ROMINA: Like for Frames 19 through 20, if I measured - we didn't get our measurements right, really measured it, like 10, like 20- you could bet on that one, because that one, we could get our measurement accurate enough.

MATT: At Frame 20 where the numbers are so close, I mean you have the exact - most of us have the exact same numbers. So, if you have the exact same numbers, the exact same distances, it's a constant speed. So, if you can take that and take the average of that and come up with your speed from there, that's a good number. And if we all get the same distance for that, we can give him that number and he can probably bet on that number. But in Frame 10, I mean it's worthless.

TIM SWEETMAN: Questions from anybody else in the room?

ROBERT SPEISER: I think it's the demand for the explanation, the demand for a reason, that pushes you through the surface features of the situation and really gets you into the heart of it. I think that one of the most important messages from this particular long-term research study has been how powerful that need for justification is, both in terms of good teaching - helping the students to build the most powerful and lasting understanding of important ideas - but also how central it is for the students themselves, how important it was for them, for the mathematics and their reasoning and their conclusions, to make sense, to be explainable, to stand up to public discussion. Their own demands for rigor turned out to be one of the findings of this study.

ROMINA: You're going from- there's exit 9, 10, 11 on the parkway. This is how we like thought about it. At exit 9, you're going 30 miles per hour. And exit 11, you're going 60. How fast where you going at exit 10? You don't know. You could have sped up like - you could have been going 60 there; you could have been going, you could have still been going 30.

JEFF: You could have sped up to 120 miles per hour for exit, for that exit 10, and slowed down to get 60. I mean, all you know is the beginning and the end. You have no clue what he did in between.

JEFF: You know, since the CEO is here now...

ROMINA: So nice of you to stop by...

CHUCK WALTER: Oh my gosh.

RALPH PANTOZZI: I was working on something of great concern to most of you. I need to go back to that meeting and I want to know is should I call off the bet? I have just a few more minutes to call off the best.

STUDENTS: Call it off.

RALPH PANTOZZI: I'd like to be convinced of why I should call it off. I could make some money here.

JEFF: But all of us are telling you that we all have different answers and we're not sure that they're right. Isn't that enough for you to call it off?

MATT: We're all taking averages; they're all approximations. We don't have exact measurements.

ROMINA: Yeah, our measurements are even off. Just right off the bat, we're just not doing good.

ROBERT SPEISER: They wanted a complete and almost total understanding of what was actually happening in this situation. That was not something that we posed to them. It was something they demanded for and of themselves. And not only did they demand it, they delivered.

RALPH PANTOZZI: Well, let me ask this then. If I call off this bet on Frame 10, I guess I need a reason to - if I could just state a reason for me to make a bet. Like, if the situation was such, I should make a bet, and if the situation is not such, I should not. Could you make a -

MATT: It depends on the motion of the cat.

VICTOR: Right, and then you have to ask them how accurate does he want his measurements anyway.

MATT: If the situation is such that the cat is moving at a constant speed, meaning that there's the same distance between frames, then you should make your bet. But if not, in the case of Frame 10 and Frame 9 to 10 and Frame 10 to 11, it jumps from 2 centimeters to 7 centimeters difference. That's a big difference. You don't know what happens in between there. So, in that case, you don't make the bet.

ROMINA: People underestimate what we can do. And if you just give us problems and we keep working at it, it builds us up; it just develops your idea. And maybe and like when we're running the world we can come up with better solutions because we know more and we can like, we've practiced it and we've been able to like have like group thinking and solutions.

[MUSIC- Kenilworth high school graduation]

NARRATOR: Eleven months later, the students who were seniors graduated from high school.

NARRATOR:†**What conditions in the mathematics classroom might be required in order to make mathematics a meaningful subject for all students, all the way through high school?**

ANKUR: Math is, I guess, part of my daily routine. Itís been there since I could remember, every day itís existed.

ROMINA: It's more than just the numbers to me. It's like you have to go deeper, you have to, if you understand something from the beginning, you're going to always understand it. You can't forget something like that.

BRIAN: It just helps me doing things, other than math, too. Like I think more in depth, very seriously about things.

MICHELLE: Now I realize, even in my high school, when I'm doing problems, I can understand a lot of times the steps I'm doing, and other people don't take those steps to look at it more analytically.

VICTOR: Nobody comes up with a solution to a problem by themselves. People have to get together and think about a problem, and get what they know, and share the information, and come to one exact solution.

BENNY: most people get stumped. But you shouldn't give up. You should always try to find methods around. You have peers. You have teachers. They're all there to help. And I found out that you can use those people as resources to help you.

JEFF: We were all different teams and it was just like the real-life setup. And our group of people had to come up with an answer. And even though we all had different answers we had to refine it to a point where everyone was happy with it. And you're going to come in contact with that no matter what you're doing.

NANCY BATON: Michael John Aiello...

[Cheering]

NANCY BATON: Ankur J. Patel...

[Cheering]

NANCY BATON: Brian Malina...

[Cheering]

NANCY BATON: Magdalena S1owolwsky... Magdalena has receives a medal for achieving the second highest average in Advanced Placement Calculus.

[Cheering]

MICHELLE: I think that, in the end, it just requires a teacher paying a lot more attention to the students, and to what every student is thinking. But I think it's worth the investment of time, because not only does it help the students with math, but it helps the students work through other problems in any subject, that the way you go about the problem, and the way you look at it and think about it can be a lot different. And it makes the student feel more important, because you think the teacher cares about you, and cares about what you're thinking.

NANCY BATON: I now declare the ceremony officially over. Congratulations to the David Brearley Class of 2000.

[Cheering]

END OF PROGRAM.