

PRIVATE UNIVERSE PROJECT IN MATHEMATICS WORKSHOPS

PROBLEMS AND POSSIBILITIES

WORKSHOP 5: BUILDING ON USEFUL IDEAS

PART 1 - THE CHANGING ROLE OF THE TEACHER: ELEMENTARY CLASSROOMS

[MUSIC]

BRIAN: ...two blues, two whites- two blues, two whites, and a blue.

ROMINA: Correct.

NARRATOR: In earlier workshops we've seen students successfully building on previous work in mathematics to tackle problems of ever-increasing complexity. In real-life classrooms it takes a long time to help students build up this solid foundation of fundamental mathematical ideas. Yet, no matter at which point the students arrive, it is possible to begin the process. In many cases, it starts with just one lesson. Let's look at some of these first activities. Later in the program we'll see what the possibilities might be.

[MUSIC]

MICHELLE DOUGHERTY: Oh I like the way Jallell is showing me he's in listening position. So is Adrian,...

NARRATOR: At the Quarles School in Englewood, New Jersey, kindergarten teacher Michelle Dougherty participated in a professional development workshop with Arthur Powell, a researcher in the Rutgers long-term study.

MICHELLE DOUGHERTY: What we're going to talk about now are the rods that you were using. I'm going to see what you learned about the rods as you were playing the game.

NARRATOR: Today, she is trying out an activity that she has never done before, one that deals with the concept of addition. Called trains, she asks

the students to find all the different ways of linking up shorter rods so that their combined length will match the length of a given longer rod.

MICHELLE DOUGHERTY: Who can raise a quiet hand and tell me one way that we could make the same size as the green rod using different colors. Adrian?

ADRIAN: Put a light green and another light green on top of the light green.

MICHELLE DOUGHERTY: How many light greens is that?

ADRIAN: Two.

MICHELLE DOUGHERTY: Can you come up and show me how you would do that? I'll give you your two light greens, and can you stand them on top of each other. But before we do that, thumbs up if you think Adrian is correct, thumbs down-- I don't think that's the same size. Jacob doesn't think so.

JACOB: I do.

MICHELLE DOUGHERTY: Most people do. Now you do. Oh, most people do. Let's see if Adrian's right. I'm going to ask you to put it right here, stand them on top of each other, and we're going to see if Adrian was correct. Is he correct.

STUDENTS: Yeah.

MICHELLE DOUGHERTY: All right, give him a hand. Nice job, Adrian. [applause] Nice job. And he knew it in his head, and then he came up and he could show us and he was right. Who knows another way we could use different colors and it will be the same size? Jacob?

JACOB: Red-- I mean, one red and two whites.

MICHELLE DOUGHERTY: One red, two whites. Thumbs up if he's correct. You don't think so, Tatiana? Is one red and two whites going to be too small or too big?

TATIANA: Too small.

MICHELLE DOUGHERTY: It's going to be too small; what do you think? Too small or too big?

STUDENT: Too small.

MICHELLE DOUGHERTY: Let's see. You said one red and--

JACOB: Two whites.

MICHELLE DOUGHERTY: Go ahead. Carefully, Andrea, can you step back, honey, give him room?

MICHELLE DOUGHERTY: They surprise me all the time. I can't believe they came up with that many ways to make the dark green rod. I didn't know if anyone would have any ideas, I didn't know if they would want to try it, you know. They're constantly surprising me with what they know.

MICHELLE DOUGHERTY: Jacob, talk to me. Stand up so everyone can see what happened.

STUDENTS: Too small. Too small.

MICHELLE DOUGHERTY: Oh no, Thalia was right. It was too small. Jacob, what do we need? What do we need to make it correct? [Suggestions from students.]

JACOB: One red.

MICHELLE DOUGHERTY: What Jacob?

JACOB: One red.

MICHELLE DOUGHERTY: Thumbs up if-- is he right with one red? Thumbs down, is he wrong. Emani you don't think he needs one red?

EMANI: No.

MICHELLE DOUGHERTY: What does he need?

EMANI: Two.

MICHELLE DOUGHERTY: He needs two reds? OK, let's see.

STUDENT: No, one, two white.

MICHELLE DOUGHERTY: You think he needs two white?

JACOB: No, I use a red.

MICHELLE DOUGHERTY: Jacob wants a red, so that's what I'm going to give him, and let's see if he's correct. Go ahead. [Jacob laughs.] Is he correct? Is that correct? Give Jacob hand. [Applause]

MICHELLE DOUGHERTY: If I continue the discussion later, I'll see how many we can get and then we can discuss how do we know when we're done-- like we discussed over the summer in the math workshop, how do we know. And I want to see what they say to that question. That will be a whole other lesson: How do we know when we're done?

ARTHUR POWELL: In terms of teaching mathematics and in terms of mathematics itself, it's very important to get students involved in looking for justifications, because at the heart of mathematics is the idea that we can look at patterns and relationships and try to understand the underlying reasons why those patterns and relationships exist. And in reasoning, in understanding why they exist one is developing ideas of proof.

MICHELLE DOUGHERTY: So how many reds do you need?

JACOB: Two reds and two whites.

MICHELLE DOUGHERTY: Thank you. Have a seat, Jacob. Jacob did it with two reds and--

STUDENTS: Two whites.

MICHELLE DOUGHERTY: I can't believe how many ways we found so far. Boys and girls, we're going to come back to this a little later-- because I know you've been sitting for a long time-- to see if we can find any more ways, or maybe-- maybe we have them all. Raise your hand if you think we have them all? We have them all. Hands down. Raise your hand if you think maybe there's one more. A couple people think there's one more. We're going to try it later;

I'll leave these here and we'll come back to it a little later because you've been sitting for awhile.

MICHELLE DOUGHERTY: When I experience the lesson first hand with the children, I seem to learn more about the children and more about how I can teach them and learn from them, because they're giving me ideas. When they say, "I don't have any more of these rods, I need more," they're giving me ideas. I need to figure out a way that they can make that size rod. So, they're constantly giving me more ideas on what I could use in my teaching.

[Music]

NARRATOR: Across town, at the Lincoln School in Englewood, several other teachers have also been expanding their classroom practice in response to their work in Arthur Power's summer workshop. Melissa Sharp is a first-year teacher, who is teaching a bilingual class of second graders.

MELISSA SHARP: What are-- you can touch the rods and figure it out. These are your rods-- all the different ways that we can make a train equal to the length of one magenta rod.

NARRATOR: In this activity, Melissa's students are also doing trains. How many trains of shorter rods can the students make that will link up to match the length of a given longer rod?

MELISSA SHARP: I was hired two weeks before the workshop started and they said, "Do you want to take a math workshop?" And I said, "Sure, I'm new; why not?" It made me think more about the importance of clarity in discussing things with students, instead of just letting them say any vague answer, really being very specific, to make sure that everybody knows exactly what that child is talking about and really having the students think about the problems they're working on, and not just simple problems that they could figure out easily, but really have them working hands-on with whatever it is that they're doing.

MELISSA SHARP: I see some of you have already started writing down when you got your paper; that's exactly what I'd like you to do, so--

NARRATOR: To help later full group discussions, Melissa and her students are using letters to keep track of their combinations. Arthur Powell continues to assist Melissa and the other teachers with math activities during the school year.

RITA: You can make another one with the red and two whites. You already have that one there. You already have that one.

MELISSA SHARP: Okay, Rita, do you think you have all the ways?

RITA: Yes, because there isn't any more smaller cubes except the lime, the red, and the whites, and I already used all the ways.

MELISSA SHARP: How do you know you used them in all the ways?

RITA: Because I tried all the different ways.

MELISSA SHARP: You tried all the different ways? Okay. So, could you use whites in any other order? I see here you have all whites. Is there any other way you would make a new train with whites?

RITA: No.

MELISSA SHARP: Is there other place that you could put the lime rods with another color rod to make one that equals the magenta rod?

STUDENT: No.

ARTHUR POWELL: I see Melissa as being exceptional in one very important way: This is her first year of teaching, the first time that she has responsibility for a class, and Melissa seems to naturally want to investigate how her students are thinking about problems. So, the questions that she asked today in class drew out from each of her students their statements as to how they were thinking about a particular problem. She was always asking her students to justify their statements.

STUDENT: Yes.

MELISSA SHARP: Why?

ARTHUR POWELL: She would ask them, "Why do you think you have them all? How do you know?" And those are questions which force students to reflect on the work that they're doing and to develop an inner criteria for knowing when they're right or wrong.

ALEXANDRA: To make a magenta with red is two red, one red with two whites, and two whites with one red, two whites and one red, so that's all the ways you could do it with red.

MELISSA SHARP: How do you know that's all the ways that you could do it with red?

ALEXANDRA: Because you put another one of these it's going to be higher, and if you take these it will be smaller. This is the only one to make this one. And this one. You put this one in the middle like this, it's going to be the same. It's the same thing but I have it.

MELISSA SHARP: Right, you already have it.

ALEXANDRA: So I need to change it.

MELISSA SHARP: It's backwards. Okay, great. I really like the way you thought about that problem. She put a lot of thought into that, didn't she. So, from that explanation, are you guys convinced that she has found all the ways?

STUDENTS: Yes.

MELISSA SHARP: Okay. Excellent.

ARTHUR POWELL: What are they using when they work on these problems? They're using their visual imagery, they're using their ability to talk about what they're doing, they're performing some actions; her students are going to have a very rich background in understanding of the additive structure of numbers. So, using facilities that they already have-- they walk into the classroom with, they then can engage in some very rich mathematical work.

STUDENT: I need to measure the yellow.

MELISSA SHARP: I want them to have the skills to figure things out on their own and not just give up. I want them to be curious about the problems and not just give any answer just for the sake of giving an answer, and I want them to have the basic skills to be able to do the things that they're going to need to do in the third grade and the fourth grade and so on. I think that's most important to me.

[MUSIC]

NARRATOR: Arthur continues his work with fourth grade teacher Blanche Young, whose students spent three sessions with the pizza problem. Two months later, Blanche is introducing a new, but mathematically related, problem, towers.

BLANCHE YOUNG: I'm going to be giving out some connect snap-cubes. One requirement about these snap-cubes. I borrowed them from almost every class in the building, so when I give you connect-cubes or snap-cubes, try not to mix them from one table to another. The assignment is going to ask you-- here's your question- to make what is called a tower. May I have one of the boxes from the top, okay? And first we need to understand the terminology. This is a tower, even though it's made with one unit, it's called a tower--

NARRATOR: Blanche asks her students to make as many different towers as they can by selecting from blocks of two colors.

BLANCHE YOUNG: Your group is going to be assigned to make as many four-tall towers as you can, but when you're done, you have to agree that you have gotten as many different arrangements as you can without exceeding the height of four connect-cubes. Is that clear?

[simultaneous conversations]

BLANCHE YOUNG: Whatever color that group is working on, you may give them these.

CHRISTINA: I'm making patterns..... See look this is a Christmas color.

RAISA: I already did all of those. See these two. Hold up- I messed up.

BLANCHE YOUNG: Do you recall what the assignment is? To make how tall?

S: [simultaneous conversations]

BLANCHE YOUNG: Four tall, okay, you're going to make them four tall.

RAISA: Ms. Young, I'm done.

BLANCHE YOUNG: How many did you make?

RAISA: I made six.

BLANCHE YOUNG: Can you make that same configuration in a different way?

CHRISTINA: Raisa, what did you do? Oh you're using that.

RAISA: Oh wait I got one.

CHRISTINA: I'm making Christmas colors.... Red, green, red, green... That's what I'm doing red, green and then green, red.

RAISA: I did green, red, red, green, and I need red. Two greens, and I need one more red.

BLANCHE YOUNG: The class is a multi-ability group class. One of the positive is that there's less stress on a particular skill. Students who may not be able to recall their addition multiplication facts are able to figure out a problem relatively easier than if they were confronted with paper and pencil and you had to solve these problems. So, if they have the opportunity to work with students of various abilities, they draw from each other. They kind of complement each other. And you could easily walk into a classroom and not be able to figure out which students are crackerjack math students and which students aren't. They kind of level the playing field, I guess is a way to put it.

STUDENT 1: One, 2,3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. We got 14.

STUDENT 2: Fifteen.

STUDENT 1: We did this already.

[simultaneous conversations]

BLANCHE YOUNG: ...Four, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. You got to show me. Every time you pick one and pair it off-- if I just picked these two, why did I select these two?

TERELL: Because the one is all like blue, and this one is all with white.

BLANCHE YOUNG: Yes. And then you started selecting the others. What are you doing to select the others?

TERELL: This one is one blue, this one has one white at the top, this one has one white in the middle, this one--

BLANCHE YOUNG: Good. All right, can you finish then? All right, let's put these back. You're going to finish-- Now, I'm having a little trouble with this one because--

TERELL: This one is white.

BLANCHE YOUNG: Okay, at the top there is--

TERELL: One blue, one white, and two middle- two whites, two blues and it's with one white-

BLANCHE YOUNG: Very good. I understand.

ROBERT: This one has three blues on top and one yellow on the bottom. This one has three yellows on the top and one blue on the bottom, and this one has one blue on top and three yellows on the bottom.

ARTHUR POWELL: And those three are different from any of the other towers that you've made?

ROBERT: These can be, like, the same. These can be doubled, because these are together.

ARTHUR POWELL: Okay.

ROBERT: But they kind of go. Hold up. One of these can go-no-no but this can go with this. No, it can't.

ARTHUR POWELL: What do you mean "can go with"? What do you mean by that?

ROBERT: They can be together.

ARTHUR POWELL: Why do you say they can be together?

ROBERT: Like, this can be put-- I've got two of the same ones. Yeah, this one can be there and this one can be here, or this one can be here, or there. It can be in two places.

ARTHUR POWELL: Combinatorics can focus their attention on using their skills of counting to develop more efficient procedures. There are always different ways of performing the counting procedures that they come up with, so it provides a very ample area for students to talk about the mathematics, to exchange ideas, and to display their mathematical thinking.

ROBERT: --on top and one blue on the bottom and one yellow in the middle.

CHRISTINA: Red, green, red, red.

RAISA: I'm just going to do the ones that are green.

BLANCHE YOUNG: What is it that we're trying to change about the way the children-- they may not even see this as math; you and I see it as math.

ARTHUR POWELL: -. Getting students to take ownership of a problem-- you can see here that they all own this problem. They're engaged in it very deeply, and they're thinking deeply about the problem. Getting them to see that they can exchange ideas among themselves and to listen carefully to each other-- and we see some of that taking place.

BLANCHE YOUNG: I do need that; I need that assistance with them taking ownership to a problem-solving type math problem because they're quick to say, "I don't know what to do. I don't understand that." And to make it meaningful, to bring it down to their level so that they can own the problem, get engaged in the problem, and work through the problem is crucial to, crucial to right now, what's needed.

[MUSIC]

NARRATOR: At the Conover Road Elementary School in Colt's Neck, New Jersey, fourth-grade teacher Amy Martino was a researcher in the Rutgers long-term study.

DREW: This equaled forty.

AMY MARTINO: What I really pulled from that experience with Rutgers was, basically, children know a lot more than we initially think that they do, and if you can tailor learning to right where the child is at in terms of their knowledge level, it makes all the difference in the world. It's very important to listen to children and this is something that we did for years. And it got to the point where I just said, "You know, I really want to be doing this full time, working with children."

AMY MARTINO: Good morning.

NARRATOR: On a typical morning, as the students arrive, Amy greets them with a warm-up question. Today's challenge is a math problem: Create as many different equations as you can in which one side of the equation is ten.

AMY MARTINO: This activity that I call "Equations Galore" is an activity that I like to do with the children. Today we picked the number ten; generate all the equations you could that would equal ten. Initially, this particular activity with equations was really meant to be about a ten or fifteen-minute warm-up activity. It really gives me a chance to sort out what needs to be sorted out at the beginning of the school day and it gets them thinking, it gets them going right away, as soon as they come through the door.

NARRATOR: Surprised by the wide variety of different responses her students have written, Amy decided to re-arrange her schedule to allow time for a class discussion.

AMY MARTINO: Boys and Girls-You know what? Wherever you are at this point in terms of writing equations, I'd like you to stop. Okay. This really was meant to be a warm-up to get you going, and it certainly did. What I'm going to ask is that we get some folks up here who maybe want to share one of their favorite equations that comes out to ten--

AMY MARTINO: Oh my goodness. Look at that one. You want to read that one, Brian?

BRIAN: One thousand plus one thousand minus two thousand plus five thousand plus five thousand minus ten thousand plus ten equals ten.

AMY MARTINO: Oh my gosh. Look at that one. Does it work?

STUDENT: Yes.

AMY MARTINO: Look at everyone. Does that work? It's really, really something. I have a question for the class before we move on. Take a look at Brian's; do you think it makes a difference whether we read his from, let's say, going from left to right, or say we started at the end and went from right to left. Okay. Do you think we'd come up with that same ten, either way?

STUDENT: Yes.

AMY MARTINO: Yes, no? Yeah, Alexandra.

ALEXANDRA: Yeah, because when you put all the numbers together you don't have to put them in a certain order to get the same number- you just get the same number. If you're doing five plus five-- I mean, like five plus two, then if you-- that's seven, but if you do two plus five it's the same thing as-- five plus two and two plus five are the same thing.

AMY MARTINO: Okay, ... Tasha?

TASHA: I think it's sort of- it would come out because if you have, like, seven plus two plus three and then you did it three plus two plus seven, it still comes out to the same answer, except it's in a different order.

AMY MARTINO: Okay. Drew?

DREW: Well, she's saying by pluses, but what about the minuses?

AMY MARTINO: Ah. Want to say something more about that? What do you mean, what about the minuses?

DREW: Because you're going to have to have more numbers to minus then five thousand from ten thousand?

AMY MARTINO: Okay, so you're not a hundred percent sure that this is going to work going both directions.

ALEX: It doesn't work.

AMY MARTINO: Alex, do you want to say something else?

ALEX: I did it. It comes out to twelve thousand and ten and that's not ten.

AMY MARTINO: All right, so we ran into a problem there. Does anybody think that they can see why-- in other words, what Alexandra said and what Tasha said made perfect sense to me. You know, when you take something like three plus five and five plus three, you do get the same thing. What happened here?

AMY MARTINO Even though they haven't learned about order of operations or any of the formal ways of representing these more complex equations, they clearly have some very, very elegant ways of thinking about equations and how it all works.

AMY MARTINO: Tasha?

TASHA: It will work with pluses, but I don't think it would work-- like, a long equation like that, because you have pluses and minuses, and when I did it backwards it equaled three thousand.

AMY MARTINO: Okay, so you're saying then that it really does make a difference which direction we start in.

AMY MARTINO: It is a lot more work to do things this way. I believe you get a lot more than the time that you're sacrificing for it; just the warm-up activity that children were doing while I was basically taking attendance and doing morning things, the kinds of discussion and discourse that came out of that, they generated all kinds of wonderful equations that equaled ten, we saw repeated subtraction, we saw mixture of operations, we saw children generating patterns that really showed that there were infinitely many possibilities

using certain operations. I mean, all that-- the power of all of that as opposed to if I had given them, say, a math worksheet to work on--

AMY MARTINO: Yes, Patrick.

PATRICK: I did it with pluses but you have to stop a certain place, like it goes 4 plus 6, 5 plus 5, 8 plus 2, but then- 9 plus 1, but then you stop there, you can't go really any farther. You have to go into minuses.

AMY MARTINO: Okay. Paul do you want to say something?

PAUL: Well you can go on with addition but you'd have to go into the negative numbers, so probably the next one would be negative one plus one- that would be ten, eleven that is- would equal ten. And you'd keep going on and on into the negatives.

AMY MARTINO: Okay. So, if we decided to go into the negatives, Paul's saying you could probably come up with more, do you agree with that Patrick?

PATRICK: Yeah.

AMY MARTINO : Yeah. Okay. Kim?

KIM: I had a pattern from division, and I started with sixty divided by six, and I got the pattern by, like, the first number of sixty is six, so I divided it by six, and then I went up to seventy, divided by seven, and I worked my way up to three hundred and ten.

AMY MARTINO: That's pretty impressive.

KIM: And when I got three hundred and ten, I divided it by thirty-one because they're the two first numbers and it equaled ten.

AMY MARTINO: That equaled ten as well. Very nice. Let me kind of pull you together for a minute, okay?

[MUSIC]

NARRATOR: What actions, taken by these teachers across the grade levels, seem to encourage students to think mathematically? In what ways are these effective?

[MUSIC]

AMY MARTINO: -- there was some question about it, and then the other side of the room did everything in reverse and tried to see what was happening.

PART 2 - PASCAL'S TRIANGLE AND HIGH SCHOOL ALGEBRA

STEPHANIE: Let's make a deal, everything we make, we have to check.

DANA: I'll always make it, and you'll always check it.

STEPHANIE: You make it and I'll check it.

NARRATOR: How can students make connections between the counting problems they do in elementary school and high school algebra? One key is Pascal's triangle, which is often included in the traditional high school algebra curriculum.

STUDENT 1: ... One, 6, 15, 20, 15, 6, 1.

STUDENT 2: All right, now I see it.

NARRATOR: When the Kenilworth Focus Group students first approached the combinatorics problems, they often began by looking for patterns.

BRIAN: Well, once when we find one, we just do the opposite of it.

ALICE ALSTON: What do you mean, "the opposite?"

NARRATOR: Some of the patterns that emerged were finding opposites, pairing each tower with an identical tower with opposite colors.

BRIAN: When found this one out, we just put two blues on the top and three whites in the middle.

ALICE ALSTON: Oh. Do they always have an opposite?

ROMINA: Yes.

BRIAN: Yeah.

NARRATOR: Another pattern is grouping, grouping towers in subsets, defined by the number of blocks of a chosen color.

BRANDON:... Three's group, and then switch those around, same thing.

NARRATOR: In the Kenilworth study, the students went on to find a further pattern, that each time the number of possible blocks increases by one, the number of possible towers doubles.

STEPHANIE: Well it goes like in a pattern, you have the 2 times 2 equals the 4; the 4 times 2 equals the 8, and the 8 times 2 equals the 16.

CAROLYN MAHER: I wonder why? If this is a pattern, what would you guess would be with Towers of 5?

STEPHANIE: If I had a guess?

CAROLYN MAHER: By noting this pattern.

STEPHANIE: ... Thirty-... yeah, thirty two.

CAROLYN MAHER: You would guess 32.

STEPHANIE: I would guess 32.

[Music]

NARRATOR: All of these patterns are represented in the array of numbers called Pascal's triangle. Blaise Pascal, a French mathematician, wrote about this symmetric triangle pattern in 1654, in the course of investigating probability problems.

To build Pascal's triangle, we start with 1. The first row of Pascal's triangle has two 1's. The second row starts with a 1 at each end, but also includes the number that is the sum of the pair of numbers directly above. We continue this pattern of starting with a 1 at each end, and adding every pair of numbers from the previous row to make every number in the new row.

[Music]

NARRATOR: Pascal was not the first to notice this array of numbers. It was discovered in China as early as 1150, A.D. and was documented before Pascal in Japan, Persia, and Germany.

At the Ferris High School in Jersey City, New Jersey, a math teacher uses Pascal's triangle, along with combinations activities, to help her students grasp difficult concepts in algebra. Gina was a researcher in the Rutgers long term study.

GINA KICZEK: ..the different combinations? OK.

GINA KICZEK: My name is Regina Kiczek. I've been teaching since 1972. I've been in Jersey City since 1980, teaching here at Ferris High School. I've changed as a teacher over time. The more I have grown accustomed to listening to what students have to say and asking questions that will elicit their thinking, the more I've come to understand that the way I see something is very rarely the way that they see it.

STUDENT: It's like a pattern.

GINA KICZEK: It's like a pattern? Okay. You discovered something about this pattern?

NARRATOR: Gina's algebra II class is mid-way through a unit that includes Pascal's triangle and the binomial theorem. The binomial theorem describes how to multiply or expand an expression such as $a + b$ to any power. These expansions are frequently used in algebra, but they quickly become tedious if one has to multiply them out by hand.

Pascal's triangle can provide a shortcut for finding the co-efficients of the terms in binomial expansions. However, Gina introduces Pascal's triangle only after the students have experienced a number of counting activities such as towers and pizzas.

GINA KICZEK: Okay, tell me about towers. How does 2 to the n work with towers?

STUDENT 1: There's two different colors.

GINA KICZEK: It's two different colors. And, what's the n ?

STUDENT 2: It could be like 5 high.

GINA KICZEK: Okay. So if it's 5 high, and it's 2 different colors then it's 2 to the ?...

STUDENT 2: Fifth Power.

GINA KICZEK: Fifth Power?...

GINA KICZEK: In the standard algebra II curriculum that we follow, the binomial theorem is basically the end of the course. And it's something that we touch upon, and basically, the students just expand $a + b$ to the second, the third, the fourth, the fifth, look for a pattern. Some students will have seen Pascal's triangle before, some students will have not. It depends on the class. But basically, the idea is get to the binomial theorem, talk about expanding binomials, and then you're done, that's the end of the year, let's review.

I think that the problems that we do-- towers, pizzas, ice cream-- I think that that helps to give them an intuitive understanding. And I think that the concrete objects and the other types of things that they've been working on, I think that gives them a really good basis for understanding why it is that this happens.

GINA KICZEK: All right. So what I'd like you to do is go to your tables, to your groups...

NARRATOR: On this day, Gina is giving her students a new challenge. If ice cream is served in bowls that can hold up to 6 scoops, how many different ways can the ice cream be served? The students start by assuming that they have no more than 1 scoop of each flavor.

The groups of students apply their previous knowledge of Pascal's triangle toward this new problem.

GINA KICZEK: So let's see, you're telling me that this 64 different choices for bowls of ice cream? And you don't have to write anything but numbers?

STUDENT: Uh-huh.

GINA KICZEK: Okay. So this is kind of valuable if it works. Can you tell me what each of these numbers stands for, in terms of bowls of ice cream?

NARRATOR: The students' previous work with pizzas and towers has led them to a central idea of Pascal's triangle, [Music] that the numbers within each row map to the subsets of possible combinations.

For example, let's look at the fourth row of Pascal's triangle. When building towers 4 high, there are 4 towers with 1 blue block, 6 towers with 2 blue blocks, and 4 towers with 3 blue blocks. Likewise, every number in every row of Pascal's triangle corresponds to the number of possibilities in a grouping of towers. Added together, the sum of the numbers in each row equals the total number of combinations possible.

GINA KICZEK: ...so that's no ice cream, what's the six?

STUDENT 1: One kind of ice cream. One flavor.

GINA KICZEK: Okay.

STUDENTS 1: Then 2, 3, 4, 5--

STUDENT 1: There's 6 -all flavors.

GINA KICZEK: So this last one stands for all six?

STUDENT 1: Mm-hhmm. All of them.

GINA KICZEK: So how many do you have altogether, then?

STUDENTS: Sixty-three.

STUDENT 2: No, because you don't count that one if that one doesn't have any, right?

STUDENT 3: 63.

NARRATOR: By not counting the empty bowl, the students come up with 63 possibilities. In the second half of the 80 minute block, Gina presents a new problem, where the order of flavors does make a difference.

STUDENT 1: The cones were delivered later in the week. The owner soon discovered that most people who order cones are particular about the order in which the scoops are stacked.

NARRATOR: The problem is, how many different ice cream cones can be made, using up to 4 scoops of ice cream, by selecting from 6 different flavors?

STUDENT 2: So we had to use, like-- we've got to use, like, these flavors and, like, we use 4 instead of 6? Right?

STUDENT 1: Yeah.

STUDENT 1: Ms. Kiczek it says for this one right here that they like- like if it's vanilla over chocolate, or chocolate over vanilla. But I don't understand it, because we're just supposed to double it or something?

GINA KICZEK: Well, I don't know. What do you think? If you had vanilla and chocolate, chocolate and vanilla, is that two different things?

STUDENT 1: It's the same thing, but you're just switching it around.

GINA KICZEK: Okay. In a bowl, does it matter?

STUDENT 1: No but in a cone-

GINA KICZEK: No, but in a cone, does it matter? To some of these people it does.

STUDENT 1: Yeah.

GINA KICZEK: To some of these people it does, okay? Let's think about three scoops now.

STUDENT 2: You can start with this one on top, and then like that-- the chocolate, the cherry, and this one. That's one cone.

GINA KICZEK: Okay.

STUDENT 2: Then you can stay with that one, but instead this one, reversed that?

GINA KICZEK: Okay.

NARRATOR: By looking at the number of possibilities or permutations with 3 scoops of 3 flavors, these students are making a solid first step to solving the problem. Extending the work to 6 flavors and including all the other possibilities-- 1 scoop, 2 scoops, and so on-- will eventually lead to the solution.

STUDENT 2: So it's kind of like when we did the towers, right? Like if you have 4 high and 4 yellow, and then you put 1 blue-- so you keep moving it down.

GINA KICZEK: That's a possibility.

GINA KICZEK: As we moved through the combinatorics unit, every problem builds on what they've done before, but there's always a new wrinkle in it. What happens, for example, today, with the cups and the cones, will-- what about this new wrinkle now with the cone-- let them think about something new, let them build on what they've already done, and then learn to express themselves, write down something, and think about it.

GINA KICZEK: Okay. What do you think about that?

STUDENT 2: Oh yeah there has to be 3 more flavors.

GINA KICZEK: Oh, I see what you're saying.

STUDENT 2: So you've got to do more-- six more, so there's 12 altogether.

GINA KICZEK: I don't know, you've got to decide that.

NARRATOR: How does an intuitive understanding of problems like towers, pizzas and ice cream help students with higher mathematical ideas?

STUDENT 2: No, you get more because then you get that group with this group, and you mix those.

STEPHANIE: All right. We went back to the beginning with the towers, and we went way back to when we were building towers, like, a long time ago.

NARRATOR: In March of 1996, mathematician Robert Speiser, of Brigham Young University, interviewed Stephanie. Stephanie linked towers 2 high to each number in the second row of Pascal's triangle.

STEPHANIE: We figured out all of them, like, from this.

ROBERT SPEISER: OK. Tell me a little more about the triangle. Okay, does this have to do with towers?

STEPHANIE: Yeah.

ROBERT SPEISER: Show me the--

STEPHANIE: It would be--

ROBERT SPEISER: So these are the towers that are 2 high-- 2 blocks high. And then how do you find the 1, the 2, and the 1?

STEPHANIE: It would be-- if you're selecting green-- it would be 1-- Well, if you're selecting blue, it would be 1 with no selections of blue, 2 with 1 selection of blue, and 1 with 1 all selections of blue. It's like the towers.

ROBERT SPEISER: It's like the way you'd organized the towers before.

STEPHANIE: Mm-hmm. Yeah.

ROBERT SPEISER: How would you organize the next row so that it makes more sense-- so that it makes the most sense for you?

STEPHANIE: Oh, to the chart--- it would be--- Wait.

ROBERT SPEISER: How did you know to write those numbers?

STEPHANIE: Because 1 goes to 1 and 1, and then 1 goes here, $1 + 1$ is 2, and 1 goes there.

ROBERT SPEISER: Oh. So you do it by adding. Ah.

STEPHANIE: Yeah. $1 + 2$ is 3, $1 + 2$ is 3, and 1 goes there. That's how you do that.

ROBERT SPEISER: Oh. So that's how you got this row?

STEPHANIE: Yes.

ROBERT SPEISER: Okay.

STEPHANIE: That's how I got it.

NARRATOR: Stephanie then showed how adding either a green block or a blue block can make towers 3 high and lead to a new row of Pascal's triangle.

ROBERT SPEISER: Did you explore why the adding works?

STEPHANIE: Its choices can be green, built onto it-- it can either have a green on top of it or a blue on top of it. And there was no one with green, blue, blue. That's why.

ROBERT SPEISER: Good. It looks to me like the others worked the same way.

STEPHANIE: Yeah, you just keep building on.

NARRATOR: This is called the additional rule of Pascal's triangle. The same addition rule applies to polynomial coefficients.

[Music]

NARRATOR: In January of 11th grade, the Focus Group of five Kenilworth students met after school to work on a problem they had never seen before: the World Series problem.

In the World Series, assuming two teams are equally matched, and the first team that wins four games wins the series, what is the probability that the World Series will be won in four, in five, in six, and in seven games?

ROMINA: Why don't we do like- you know how we do like write out the blues-

CAROLYN MAHER: We'll leave you alone.

JEFF: Yeah, that's what I'm saying-

ROMINA: So that you can go all 7, because if you go all 4, it's only A, A, A. A, A, A, A, and B, B, B, B. Team A and Team B? Those are the only possibilities for four.

GINA KICZEK: We had worked with the students on a lot of different combinatorics problems, in towers and pizza, and extensions of those things. So we decided to see what was possible. Given all of the different ideas that they had built, we wanted to see if they could solve a particular probability problem without having been taught how to do it, without any formal rules or notation or anything being imposed. We just wanted to see what would happen.

ROMINA: So in 4 games, it would be like 1/2 of a chance? Or would we have to write out, with using all 7?

JEFF: See, I think that it's the hardest doing it in 4 games. Definitely hardest. So that wouldn't be one half.

BRIAN: Wouldn't it be the odds of winning 1 game, times odds of winning one game, times odds of winning game, times odds of winning one game?

JEFF: That's what I'm thinking.

ANKUR: It's a 50 percent chance of winning the first game.

BRIAN: All right. So it's like a half times a half-

GINA KICZEK: They did the problem in about an hour, and they did it correctly, and I've been studying the tape for about two years. There's a lot of mathematics on the tape. And I'm looking at not only what they did to solve the problem, but I'm trying to look for the origins of those ideas.

BRIAN: Just remember, the odds get harder to win 2 in a row, like a coin flip.

ROMINA: Yeah, that's how you do it. Half times half times half times half.

NARRATOR: Their answer, $1/16$, was added to another $1/16$ to account for both teams.

ROMINA: Would we do that for 5 games? That would be-- Yeah, there's going to be a lot.

NARRATOR: Mike worked on his own, using Pascal's triangle, while the other students worked together.

ROMINA: Would it be, like, say, the probability of something, and then it would be like B, B, B, B. And any ones that have B, B, B, B--

JEFF: Yeah, then that would be that number and that number. That's what I was thinking.

ANKUR: So we've got to do it like that.

NARRATOR: Moving on to 5 games, Romina proposed writing out all combinations, using strings of A's and B's to represent the wins.

ROMINA: Yeah, I know. I'm just saying, like, each time we look over, like, five, well, we'll see how many. You know?

GINA KICZEK: Of course, for a 4 game series, it's pretty easy. You either have 4 wins in a row for this team or 4 wins in a row for that team. And for a 5 game series, it was a little bit more complicated, and they realized that they got 8 different strings. But when they tried to figure out what the probability of that was, they knew it was 8 over something, and it was the 8 over something part that they had a little trouble with.

ANKUR: They have 8 ways of winning, but it would be

over--

JEFF: Oh, 8 over 1-- No, how do we find out?

ANKUR: Be over the total possibilities of 2^5 - 2 colors and 5 things.

GINA KICZEK: They seemed to have the idea that probability is the number of favorable outcomes over the number of total outcomes, although they never said that, they never had that definition. But it was an intuitive type of thing that they seem to have been doing.

ANKUR: Know what I'm talking about or no?

JEFF: Yeah, it's got to be over 2. The total possibility's 4 spaces.

ANKUR: Yeah, 4 spaces.

JEFF: Yeah, all right, it makes sense-- And that would be 8 over 2 to the fifth, do you think?

ANKUR: That's 16.

JEFF: And then 8 over 2 to the fifth?

ANKUR: I guess.

JEFF: Which would be 32.

MIKE: Is there's 32 possibilities for 5 games.

JEFF: Yeah. That sounds--

ANKUR: I think there's more.

BRIAN: For how many games?

JEFF: Five.

ROMINA: Hold on. You've got 8?

JEFF: 5 spaces.

ANKUR: Total possibilities.

JEFF: 32 for 5.

GINA KICZEK: So then they got to a 6 game series. That was a little bit more difficult to list all the different possibilities for 6 games, but they did it. When they got to the 7 game series, they realized that that was going to be a lot to count.

JEFF: You see doubles in that? I can't even look at it.

ROMINA: You want me to read them?

ANKUR: For 7?

ROMINA: With "A" winning.

ANKUR: Did you just randomly write them, or did you do them in some order?

ROMINA: I started in some order, then I-- It's hard though, because you're just, like-- I don't know. Did you write them all out?

ANKUR: I wrote them out.

ROMINA: Oh, you did?

ANKUR: I wrote out 10.

NARRATOR: Ankur found out that his winning probabilities for 4, 5, 6, and 7 games added up to 1.

ANKUR: It is right. 40 out of 128. The whole thing adds up to 1.

BRIAN: Do they match with them?

ANKUR: They match.

JEFF: Wait, 40 out of 128?

ANKUR: Yeah, it works.

GINA KICZEK: They looked at it in cases-- 4 game, 5 game, 7 game series. They got the probability of each one individually. They saw that they gave them a total of 1. They knew that that was supposed to happen. And they were

ready to present their solution, all using representations that were basically retrieved from earlier investigations, and maybe modified a little bit to fit the situation.

JEFF: So basically, what we did was, that could be 2 possibilities, that could be 2 possibilities, that could be 2, that could be 2. And that was like where we went back to the old days, and it was like 2 to the n. So $2 \times 2 \times 2 \times 2$. That's how we got 16. And that would be the bottom number. And then in order win the 4 games, these have to be either all A's or all B's. So we got 2 out of 16, for winning at 4 games, which is probability of winning in 4 games. That make sense?

MIKE: They have something that works for that first one, but does it work for-

JEFF: Yeah. We're going to go on. So for the next one, we're going to do the same situation, but this would be 2 to the 5th. So that's going to be out of 32. And 32's the bottom number. And then, I think for these we were just kind of-- we went through them. That's why there are strings of A's and B's on everyone's paper. In order to get these, we went through all the possibilities where there was 5, 5 places, and A or B was in 4 of them. And we went through all of them, and that's how we got that. And then we ended up with 8 of 32 put for that. Now that's not too convincing, because we just went through them. But we went through all the ones that were out of 5, with 4 A's. And that's how we got that. I don't think we have a real, concrete mathematical backing to that.

NARRATOR: At this moment, Mike presented his approach. Mike used Pascal's triangle to explain his strategy.

MIKE: I just found, like you take the fourth number of each one. For some reason if you double each number, because you have 2 teams, you get the possibilities for 4 games, 4 games- equals two, right? You've got 8, 20, and 40 like they said. Those last- those 3 games that they won, the first 3 games, if they win that, that would be like there's 3 possibilities- would be- if they win the next game- or if they win- I don't know how to explain this. On the third game...I don't know.

JEFF: I guess if we were going to say-- if was out of 8 games, then there would be 35? The probability would be 35 out of-- you know what I'm saying?

ANKUR: Yeah.

BRIAN: Yeah.

MIKE: It would be 1, 7--

ANKUR: Just add the 15 and 20 for 35.

JEFF: So I mean, there's got to be something there, because it wouldn't all-

MIKE: It would be 35 doubled.

ANKUR: Yeah.

JEFF: Yeah. 35 for one team.

MIKE: But the limits of the problem are you have to win 4 out of 7. Not 4 out of 8.

JEFF: Oh yeah, I know.

GINA KICZEK: So Michael notices in the triangle that on one of the diagonals, he finds the numbers 1, 4, 10, and 20. And the counts, in each case, the count for a 4 game series, the number of ways you can win a series in 4 games was 2, and the number of ways you could win it in 5 games was 8, and in 6 games was 20, and in 7 games was 40. So he's got 1, 4, 10, and 20 in this diagonal. And if you double them, he said that that's 2, 8, 20, and 40. "So there's obviously some connection," he said. "But I don't know what it is yet."

So they spent some time looking at that connection. I think initially it's just an interesting insight on Michael's part. He notices a pattern there. And noticing that connection spurred all kinds of activities over the next three sessions. That allowed them to do some pretty sophisticated mathematics.

NARRATOR: In May of their junior year, Kenilworth High School students returned to school one evening around 7:30 p.m. for a research session with Carolyn Maher and her colleagues from Rutgers University.

Carolyn began the session by asking them to review what they had discussed in their pre-calculus class earlier that day.

The class had touched on binomial expansions, and the students had learned about a way to calculate the co-efficient of any term without having to write out Pascal's triangles. The notation is called N choose R . It evaluates how many ways there are of choosing R objects from a set of N objects.

Mike drew Pascal's triangle, and explained how the numbers could be assigned to the N choose R notation.

MIKE: All right. This would be like 3 choose 1. How many different places could you put that 1, that one guy- there's only one place. The next one would be 3 choose 3. There's obviously 3 different places-

CAROLYN MAHER: You 3 choose what? What's the next one?

MIKE: -3 choose 1. The next would be 3 choose 2, which you just figured that out-- is 3. And the last one is 3 choose 3. You can only put those 3 people in 3 places. You can't-- no other place to put them.

CAROLYN MAHER: I have another question. You could write more rows of that triangle. And now you're telling me you can write them as the "choose" way, you've called that. So can you take, let's say, another row or two? And show me the addition rule, and what it looks like, with your new notation for a particular row.

MIKE: Add this and this, and go like that?

CAROLYN MAHER: Sure. Or 3 and 3 is 6. Show me what that looks like with that new notation.

MIKE: All right. Let's go to this one. This would be, like, 3 different places, I guess.

JEFF: Which one are we looking at?

MIKE: That one right there.

JEFF: That would be a plus b to the third?

MIKE: Let's say you have-- like, here's a number, right? Zero means no toppings, 1 would be a topping. So the first category is everything with no toppings, and that's your number for that one. This is like binary numbers or something, Next would be- there's all the ones that have 1 topping.

JEFF: Mike, you got to make that a zero at the end. You messed up.

MIKE: What? I knew that. There's your 3 choose 1, and there's 3 different combinations you can put that. And I can go on forever doing this. But when have a new- when you add another place, another topping-

JEFF: That can be one or the other- one or the other- one or the other-

MIKE: So it could be one or the other. It could be a zero or a one, a zero or a one, a zero or a one. So all these 3's would either move up a step onto the next category, and have 2 toppings, or they might stay behind and still only have 1, if they have the zero. So 3, I get a topping-- go to this one. And 3 won't- will stay. And obviously, this guy's going to get a topping; that's why you add this one. So now this guy's going to have-- without toppings-- you're going to add a topping onto him-- and it's going to be 1 topping. These 3 with 1 topping won't get one. So, you know, you can put them in the same category as this one, that's 4.

JEFF: Yeah. Those are 4.

MIKE: And, you know, the 3 that had 2 toppings won't get any.

JEFF: Yeah. So that'll go to the left?

MIKE: And you'll put them together with the ones that did get some. That's why you would add- keep on adding.

CAROLYN MAHER: Well I want you to show me how the addition rule works, in general.

JEFF: $N \text{ choose } X \text{ plus } N \text{ choose } X + 1$

MIKE: -Equals that

JEFF: -plus 1 equals that right there. Well that's because this would be gaining an X and going into the $X + 1$, and this would be losing an X .

MIKE: No, no, no-

ANKUR: That stays the same.

JEFF: That's staying the same, and that's- is the $X + 1$

MIKE: And the toppings going to change because you have more-

JEFF: -because you have more things. And why do it? -Because when you add another topping on to it, say the toppings were one and zero, if it gets a topping, that's why it goes up to the $X + 1$, and since it doesn't get anything, it will stay the same. And in this one, it's staying the same, right? And that's why it's going there, like saying that's the zero, and going to there. Make sense?

BRIAN: Yes, it actually does.

JEFF: So that would be the general addition rule, in this case.

CAROLYN MAHER: In fact, I wish someone would do it on the board on the right there, write that addition statement, using factorial notations.

JEFF: Minus X plus- exactly. You know like, how intimidating this equation must be, like if you just pick up a book and look at that?

CAROLYN MAHER: Could you very carefully check that arithmetic?

MIKE: You think we're wrong?

ANKUR: Yeah, it's right there.

JEFF: Where is it?

ANKUR: It's right above n over x .

MIKE: There you go.

CAROLYN MAHER: You sure?

MIKE: Yeah, I'm sure. You got anything else?

NARRATOR: How do the Pizza problems, Towers problems, and World Series problem relate to Pascal's Triangle?